Sharp bounds on the critical stability radius for relativistic charged spheres

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This work is dedicated to the memory of my father Dan Andréasson (1933-2008).

Abstract

In a recent paper by Giuliani and Rothman [16], the problem of finding a lower bound on the radius R of a charged sphere with mass M and charge $Q < M$ is addressed. Such a bound is referred to as the critical stability radius. Equivalently, it can be formulated as the problem of finding an upper bound on M for given radius and charge. This problem has resulted in a number of papers in recent years but neither a transparent nor a general inequality similar to the case without charge, i.e., $M \leq 4R/9$, has been found. In this paper we derive the surprisingly transparent inequality

$$\sqrt{M} \leq \frac{\sqrt{R}}{3} + \sqrt{\frac{R}{9} + \frac{Q^2}{3R}}.$$

The inequality is shown to hold for any solution which satisfies $p + 2p_T \leq \rho$, where $p \geq 0$ and p_T are the radial- and tangential pressures respectively and $\rho \geq 0$ is the energy density. In addition we show that the inequality is sharp, in particular we show that sharpness is attained by infinitely thin shell solutions.

1 Introduction

Black holes for which the charge or angular momentum parameter equals the mass are called extremal black holes. They are very central in black hole

thermodynamics due to their vanishing surface gravity and they represent the absolute zero state of black hole physics. It is quite generally believed that extremal black holes are disallowed by nature but a proof is missing. One possibility to obtain an extremal black hole is to produce one from the collapse of an already extremal object. Previous mainly numerical studies ([12], [7]) have concluded that when $Q < M$ collapse always takes place at a critical radius R_c outside the outer horizon, and as Q approaches M, this value approaches the horizon. This is similar to the non-charged case where the Buchdahl inequality implies that collapse will take place when $R < 9M/4$, i.e., $R_c = 9M/4$, cf. [10]. In the charged case the critical value is expected to be smaller due to the Coulomb repulsion, and this is also shown to be the case below and in particular as $Q \to M$ the stability radius does approach the outer horizon. For more information on the relation of this topic to extremal black holes and black hole thermodynamics we refer to [7], [16], [13] and [11] and the references therein.

The problem of finding a similar bound as the classical Buchdahl bound for charged objects have resulted in several papers; some of these are analytical, cf. [16], [19], [13], [14], [17] and [20], whereas others are numerical or use a mix of numerical and analytical arguments, cf. [7], [12], and [15] to mention some of them. We refer the reader to the sources for the details of these studies but in none of them a transparent bound has been obtained (except in very special cases), on the contrary they have been quite involved and implicit. Moreover, most of these studies rely on the assumptions made by Buchdal, i.e., the energy density is assumed to be non-increasing and the pressure to be isotropic.

In this work we will show that

$$\sqrt{m_g} \leq \frac{\sqrt{r}}{3} + \sqrt{\frac{r}{9} + \frac{q^2}{3r}}, \tag{1}$$

given that $q < r$ (which is a physically natural assumption, cf. the discussion below), and that $p + 2p_T \leq \rho$, where $p \geq 0$ and p_T are the radial- and tangential pressures respectively and $\rho \geq 0$ is the energy density. Here we have used lower case letters m_g, q and r to stress that the inequality holds anywhere inside the object. We refer to the equations (3) and (9) below for the exact definitions of these quantities. To the best of our knowledge this bound has not appeared in the literature before.

In the non-charged case a general proof of the Buchdahl inequality $2m/r \leq 8/9$, in the case when $p + 2p_T \leq \rho$, was first given in [1]. A completely different proof was then given by Stalker and Karageorgis [18] where also several other situations were considered, e.g. the isotropic case

where $p = p_T$. The advantage of the method in [18] (which is related to the method by Bondi [8] which however is non-rigorous) compared to the method in [1] is that it is shorter and that it is more flexible in the sense that other assumptions than $p + 2p_T \leq \rho$ can be treated. On the other hand the result in [18] is weaker than the result in [1] in the sense that the latter method implies that the steady state that saturates the inequality is unique, it is an infinitely thin shell. Indeed, in [1] it is shown that given any steady state, the value of $2m/r$ for this state is strictly less than the value $2m/r$ of a state for which the matter has been slightly re-distributed and this monotonic property continues until an infinitely thin shell has been reached for which $2m/r = 8/9$. The method in [18] also shows sharpness but only in the sense that there are steady states with $2m/r$ arbitrary close to $8/9$, leaving open the possibility that different kinds of steady states might share this feature. Moreover, since the assumption $p + 2p_T \leq \rho$ is satisfied by solutions of the Einstein-Vlasov system it is natural to ask if there exist regular static solutions to the *coupled* system which can have $2m/r$ arbitrary close to $8/9$. This question is given an affirmative answer in [2], where in particular it is shown that arbitrary thin shells which are regular solutions of the spherically symmetric Einstein-Vlasov system do exist. On the contrary, the matter quantities and the corresponding spacetimes constructed in [18] for showing sharpness cannot be realized by regular solutions of the Einstein-Vlasov system. The construction in [18] gives that a solution which nearly saturates the inequality $2m/r \leq 8/9$ satisfies $p + 2p_T = \rho$, and in addition p_T and ρ are discontinuous. Neither of these two properties can be realized by regular solutions of the (massive) Einstein-Vlasov system.

In the present work where we study charged objects we will adapt the method in [18] to show the inequality (1) and its sharpness. This again supports the claim above that this method is very flexible. We have not been able to carry out the strategy in [1] in this case. If we have succeeded it would have given a more complete characterization, cf. the discussion above. However, we do show in Theorem 2 below that an infinitely thin shell solution (with properties specified in the theorem) saturates the inequality, although we cannot show that no other steady states can saturate it as well. We also mention that in [5] a numerical study of the coupled Einstein-Maxwell-Vlasov system is carried out which supports that there are arbitrarily thin shell solutions for this system which saturate the inequality (1).

The outline of the paper is as follows. In the next section the Einstein equations will be given and some basic quantities will be introduced. In section 3 the main results are stated and section 4 is devoted to the proofs. In the final section we discuss our inequality in view of the bound derived

in [16] for a constant energy density profile.

2 The Einstein equations

We follow closely the set up in [16] but here we also allow the pressure to be anisotropic, i.e., the radial pressure p and the tangential pressure p_T need not be equal. We assume throughout the paper that p, the energy density ρ, and the charge density j^0 are non-negative. We study spherically symmetric mass and charge distributions and we write the metric in the form

$$ds^2 = -e^{2\mu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2),$$

where $r \geq 0$, $\theta \in [0, \pi]$, $\varphi \in [0, 2\pi]$. It is well-known that the Reissner-Nordström solution for the charged spherically symmetric case gives

$$e^{-2\lambda(r)} = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} = e^{2\mu(r)},\ r \geq R. \tag{2}$$

Here R is the outer radius of the sphere and Q is the total charge. This solution is a vacuum solution. The purpose of this work is to investigate the behaviour of λ and μ when the matter and charge densities are non-zero for $r < R$. Before writing down the Einstein equations let us introduce some quantities following [16]. Let

$$q(r) = 4\pi \int_0^r e^{(\lambda+\mu)(\eta)}\, \eta^2 j^0\, d\eta, \tag{3}$$

and

$$m_i(r) = 4\pi \int_0^r \eta^2 \rho\, d\eta, \tag{4}$$

where $q(r)$ is the charge within the sphere with area radius r and $m_i(r)$ is the mass within this sphere. The subscript i is used to distinguish m_i from the gravitational mass m_g which is defined below. Let us also introduce the quantity

$$F(r) = \int_0^r \frac{q^2(\eta)}{\eta^2} d\eta.$$

The Einstein equations for λ and μ now read (cf. [7] and [16])

$$\frac{1}{r^2} + \frac{2\lambda_r e^{-2\lambda}}{r} - \frac{e^{-2\lambda}}{r^2} = 8\pi\rho + \frac{q^2(r)}{r^4}, \tag{5}$$

and

$$\frac{1}{r^2} - \frac{2\mu_r e^{-2\lambda}}{r} - \frac{e^{-2\lambda}}{r^2} = -8\pi p + \frac{q^2(r)}{r^4}, \tag{6}$$

where the subscript r denotes differentiation with respect to r. Equation (5) can be written as

$$\frac{d(e^{-2\lambda}r)}{dr} = 1 - 8\pi r^2\rho - \frac{q^2(r)}{r^2}, \tag{7}$$

so that

$$e^{-2\lambda} = 1 - \frac{2m_i(r)}{r} - \frac{F(r)}{r}. \tag{8}$$

By requiring that (8) matches the exterior solution (2) at $r = R$ gives

$$1 - \frac{2M}{R} + \frac{Q^2}{R^2} = 1 - \frac{1}{R}\int_0^R (8\pi\rho\eta^2 + \frac{q^2}{\eta^2})d\eta$$

or

$$M = \frac{1}{2}\int_0^R (8\pi\rho\eta^2 + \frac{q^2}{\eta^2})d\eta + \frac{Q^2}{2R},$$

which defines the total gravitational mass M. In view of this relation we now define the gravitational mass m_g within a given area radius r by

$$m_g(r) = m_i(r) + \frac{F(r)}{2} + \frac{q^2(r)}{2r}. \tag{9}$$

In terms of the gravitational mass we thus get

$$e^{-2\lambda(r)} = 1 - \frac{2m_g(r)}{r} + \frac{q^2(r)}{r^2}. \tag{10}$$

Let us also write down the Tolman-Oppenheimer-Volkov equation which follows from the Einstein equations, cf. [7], but note that in our case p is allowed to be different from p_T which modifies the equation accordingly

$$p_r = \frac{qq_r}{4\pi r^4} + \frac{2}{r}(p_T - p) - (\rho + p)e^{2\lambda}(\frac{m_g(r)}{r^2} + 4\pi rp - \frac{q^2}{r^3}). \tag{11}$$

3 Set up and main results

The problem of finding an upper bound on the total gravitational mass that a sphere of area radius R with total charge Q can hold, or equivalently, to find the smallest radius R_c, referred to as the critical stability radius, for which a physically acceptable solution of the Einstein equations can be found, is formulated in [16] as follows:

A physically acceptable solution should satisfy

$$\rho \geq 0,\ p \geq 0 \ \text{ and } \ \mu > -\infty, \tag{12}$$

$$0 \leq Q < M,\ R > R_+. \tag{13}$$

Here $R_+ = M + \sqrt{M^2 - Q^2}$ is the outer horizon of a Reissner-Nordström black hole. The quantities m_g and q should satisfy

$$m_g(R) = M,\ q(R) = Q, \tag{14}$$

$$q \leq m_g,\ m_g + \sqrt{m_g^2 - q^2} < r. \tag{15}$$

We see immediately that these relations imply that $q/r \leq m_g/r < 1$. We will in addition assume that the following condition holds

$$p + 2p_T \leq \rho. \tag{16}$$

The condition (16) is likely to be satisfied for most realistic matter models, cf. [9], and in particular it holds for Vlasov matter, cf. [4] for more information on this matter model.
Remark. In [1] and [3] the following generalization of this condition was imposed, namely that

$$p + 2p_T \leq \Omega\rho,\ \text{for some } \Omega \geq 0. \tag{17}$$

However, in contrast to the non-charged case where a bound on M is given by a simple formula depending on Ω the simplicity is completely lost in the charged cased except when $\Omega = 1$. Now, the case $\Omega = 1$ should be considered as the principal case, cf. [9], and in the non-charged case it is when $\Omega = 1$ that the classical bound $2m/r < 8/9$ is recovered.

We are now almost ready to state our main result but first we define what we mean by a regular solution of the spherically symmetric Einstein equations. We say that $\Psi := (\mu, \lambda, \rho, p, p_T, j^0)$ is a regular solution if the matter quantities ρ, p, p_T and j^0 are bounded everywhere and C^1 except possibly at finitely many points, p has compact support and the equations (3), (5), (6) and (11) are satisfied (where the matter quantities are C^1) and the constraints (12) and (15) are satisfied.

Theorem 1 *Let Ψ be a regular solution of the Einstein equations and assume that (16) holds. Then*

$$\sqrt{m_g(r)} \leq \frac{\sqrt{r}}{3} + \sqrt{\frac{r}{9} + \frac{q^2(r)}{3r}}. \tag{18}$$

Moreover, the inequality is sharp in the subclass of regular solutions for which $p_T \geq 0$.

Let us immediately make a consistency check so that (18) ensures that the stability radius is strictly outside the outer horizon. Thus we wish to show that the inequality (18) implies that $e^{-2\lambda(r)} = 1 - \frac{2m_g}{r} + \frac{q^2}{r^2} > 0$, or equivalently that

$$\sqrt{\frac{m_g}{r}} < \sqrt{\frac{1}{2} + \frac{q^2}{2r^2}}.$$

In view of inequality (18) this holds if

$$\frac{1}{3} + \sqrt{\frac{1}{9} + \frac{q^2}{3r^2}} < \sqrt{\frac{1}{2} + \frac{q^2}{2r^2}}.$$

An elementary computation shows that this is true as long as

$$\frac{q^2}{r^2} < 1,$$

which always holds.

The proof of Theorem 1 relies on the method in [18] for the non-charged case. In the introduction we discussed the strength of this method but also its shortages; the question of uniqueness of the steady state that saturates the inequality (18) is left open, and the constructed steady states that nearly saturate the inequality cannot be a solutions of the coupled Einstein-Vlasov system (these issues were answered in [1] and [2] respectively). Furthermore, it is not completely obvious from the construction in [18] that these solutions approach an infinitely thin shell. This point also carries over in our proof of sharpness in Theorem 1 and we therefore find it natural to include a proof of the fact that an infinitely thin shell does saturate (18), although we have not been able to adapt the strategy in [1] to show that no other steady state can have this property. Furthermore, the numerical study in [5] supports that the maximizer for the spherically symmetric Einstein-Vlasov-Maxwell system is an infinitely thin shell.

We therefore investigate a sequence of regular shell solutions which approach an infinitely thin shell and adapt the method in [3]. More precisely, let Ψ_k be a sequence of regular solutions such that $p_k, (p_T)_k, j_k^0$ and ρ_k have support in $[R_k, R]$. Denote by M_k the total gravitational mass and by Q_k the total charge of the corresponding solution in the sequence and assume that $Q := \lim_{k\to\infty} Q_k$, and $M = \lim_{k\to\infty} M_k$ exist and that $\sup_k q_k/r < 1$. Furthermore, assume that $\int_{R_k}^R r^2 p_k dr \to 0$, and $\int_{R_k}^R r^2(2(p_T)_k - \rho_k)\, dr \to 0$ as $k \to \infty$.

Theorem 2 *Assume that* $\{\Psi_k\}_{k=1}^{\infty}$ *is a sequence of regular solutions with support in* $[R_k, R]$ *with the properties specified above and assume that*

$$\lim_{k\to\infty} \frac{R_k}{R} = 1. \tag{19}$$

Then

$$\sqrt{M} = \frac{\sqrt{R}}{3} + \sqrt{\frac{R}{9} + \frac{Q^2}{3R}}. \tag{20}$$

Remark. That sequences exist with these properties, in particular the property (19), has been proved for the (non-charged) Einstein-Vlasov system, cf. [2] (and [6] for a numerical study). The investigation carried out in [5] also supports that such sequences exist for the Einstein-Vlasov-Maxwell system.

4 Proofs

Proof of Theorem 1. As described above our method of proof is an adaption of the method in [18] to the charged case. Let a regular solution be given and let us define

$$m_\lambda(r) = m_i(r) + \frac{F(r)}{2} = m_g - \frac{q^2}{2r}, \tag{21}$$

and let

$$x \equiv \frac{2m_\lambda}{r}, \;\; y \equiv 8\pi r^2 p, \;\; z \equiv \frac{q^2}{r^2}.$$

Note that the conditions (12) and (15) imply that

$$x < 1, \; y \geq 0, \; \text{and } z < 1. \tag{22}$$

Indeed, the two latter bounds are immediate and the former follows since (15) gives that $q^2 > m_g^2 - (r - m_g)^2$ so that

$$x = \frac{2m_\lambda}{r} = \frac{2m_g^2}{r} - \frac{q^2}{r^2} < \frac{2m_g}{r} - \frac{m_g^2 - (r - m_g)^2}{r^2} = 1. \tag{23}$$

Lemma 1 *The variables* (x, y, z) *give rise to a parametric curve in* $[0, 1) \times [0, \infty) \times [0, 1)$ *and satisfy the equations*

$$8\pi r^2 \rho = 2\dot{x} + x - z, \tag{24}$$

$$8\pi r^2 p = y, \tag{25}$$

$$8\pi r^2 p_T = \frac{x + y - z}{2(1 - x)}\dot{x} + \dot{y} - \dot{z} - z + \frac{(x + y - z)^2}{4(1 - x)}, \tag{26}$$

where the dots denote derivatives with respect to $\beta := 2 \log r$.

Proof of Lemma 1: The proof is a straightforward computation using the Einstein equations (5) and (6) and the Tolman-Oppenheimer-Volkov equation (11).

□

Now let

$$w(x,y,z) = \frac{(3(1-x)+1+y-z)^2}{1-x}.$$

Differentiating with respect to β gives

$$\dot{w} = \frac{4-3x+y-z}{(1-x)^2}\big[(3x-2+y-z)\dot{x} + 2(1-x)\dot{y} + 2(1-x)\dot{z}\big]. \tag{27}$$

Now, using the expressions of the matter terms given in Lemma 1 the condition $p + 2p_T \leq \rho$ can be written

$$(3x-2+y-z)\dot{x} + 2(1-x)(\dot{y}-\dot{z}) \leq \frac{-\alpha(x,y,z)}{2}, \tag{28}$$

where

$$\alpha = 3x^2 - 2x + (y-z)^2 + 2(y-z).$$

From (27) and (28) it now follows that

$$\begin{aligned} \dot{w} &= \frac{4-3x+y-z}{(1-x)^2}\big[(3x-2+y-z)\dot{x} + 2(1-x)\dot{y} + 2(1-x)\dot{z}\big] \\ &\leq -\frac{4-3x+y-z}{2(1-x)^2}\,\alpha(x,y,z). \end{aligned} \tag{29}$$

Since $0 \leq x < 1, \quad y \geq 0$ and $0 \leq z < 1$, it follows that w is decreasing whenever $\alpha > 0$, which implies that

$$w \leq \max_E w(x,y,z), \tag{30}$$

where

$$E = \{(x,y,z) : 0 \leq x \leq 1,\ y \geq 0,\ 0 \leq z \leq 1 \text{ and } \alpha(x,y,z) \leq 0\}.$$

To solve this optimization problem we introduce $s = y - z$ and note that $\max_E w(x,y,z) = \max_{E'} w(x,s)$ where

$$E' = \{x,s) : 0 \leq x \leq 1,\ s \geq -1 \text{ and } \alpha(x,s) \leq 0\}.$$

It is straightforward to conclude that there are no stationary points in the interior of E', so the maximum is attained at the boundary $\partial E'$ of E'. The Lagrange multiplier method leads to the following system of equations

$$(1-x)(6(1+s)+4(3x-1))+2(1+s)^2=0, \tag{31}$$
$$x(3x-2)+s(s+2)=0. \tag{32}$$

From (32) we have that $s^2=-2s-x(3x-2)$ which substituted into (31) results in the equation

$$(x+s)(1-x)=0. \tag{33}$$

If $x=-s$ we get from (32) that $4s^2+4s=0$ so that either $s=0=x$ or $s=-1=-x$. In the latter case we get $w(1,-1)=0$, and the former case gives $w(0,0)=16$. We thus conclude that $w\leq 16$ throughout the curve. Since $p\geq 0$ it follows from the inequality $w\leq 16$ that

$$\big(3(1-\frac{2m_g}{r}+\frac{q^2}{r^2})+1-\frac{q^2}{r^2}\big)^2\leq 16\big(1-\frac{2m_g}{r}+\frac{q^2}{r^2}\big). \tag{34}$$

This is easily seen to be equivalent to

$$\big(\frac{6m_g}{r}-\frac{2q^2}{r^2}\big)^2\leq\frac{16m_g}{r}. \tag{35}$$

Taking the square root of both sides and rearranging leads to

$$\big(\sqrt{m_g}-\frac{\sqrt{r}}{3}-\sqrt{\frac{r}{9}+\frac{q^2}{3r}}\big)\big(\sqrt{m_g}-\frac{\sqrt{r}}{3}+\sqrt{\frac{r}{9}+\frac{q^2}{3r}}\big)\leq 0. \tag{36}$$

Since the second bracket is always non-negative and vanishes only if $m_g=q=0$ we have

$$\sqrt{m_g}-\frac{\sqrt{r}}{3}-\sqrt{\frac{r}{9}+\frac{q^2}{3r}}\leq 0, \tag{37}$$

which is the first claim.

To show sharpness we will construct a spacetime such that the corresponding curve from Lemma 1 intersects a small neighbourhood of $(x_q,0,z_q)$, where $z_q<1$ is a given ratio q^2/r^2 and x_q is the corresponding value of x when equality holds in (37), i.e.,

$$x_q:=\frac{4}{9}-\frac{z_q}{3}+\frac{4}{3}\sqrt{\frac{1}{9}+\frac{z_q}{3}}.$$

We will construct such a spacetime by showing that there exists a curve

$$x=x(\tau),\ \ y=y(\tau),\ \ z=z(\tau);\ \ \tau\in[0,\infty),$$

which passes near $(x_q,0,z_q)$ and in addition has the following properties

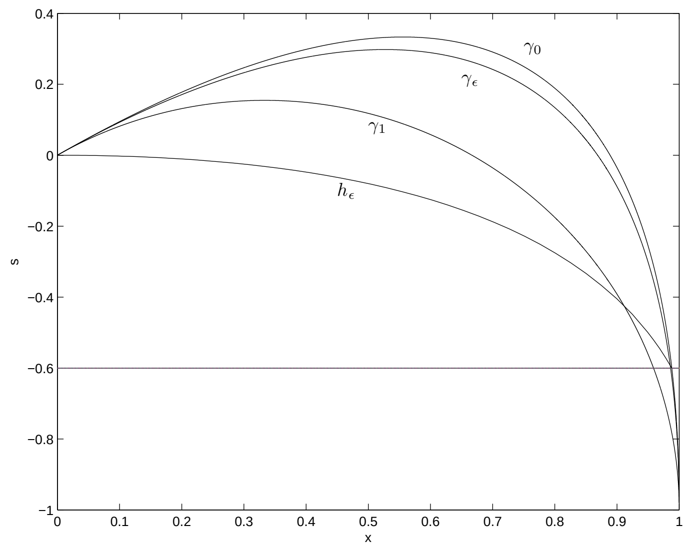

Figure 1: The curves γ_0, γ_1, γ_ϵ and h_ϵ.

- (A1) $\frac{1}{\alpha}\frac{dw}{d\tau}$ is negative and locally integrable,
- (A2) x(0)=y(0)=z(0)=0,
- (A3) $0 \le x(\tau) < x_q,\ z(\tau) \le z_q$,
- (A4) $y(\tau) = 0$ for all large enough τ, $x(\tau) \to 0$ and $z(\tau) \to 0$ as $\tau \to \infty$,
- (A5) the curve is C^1 except for finitely many points.

Below we will denote $s = y - z$ as above and the curves $(x(\tau), y(\tau), z(\tau)) \in [0,1) \times [0,\infty) \times [0,1)$, and $(x(\tau), y(\tau), s(\tau)) \in [0,1) \times [0,\infty) \times (-1,\infty)$ will be used interchangeably. Let us first see that if we have a curve which satisfies (A1)-(A5) a spacetime can be constructed. Indeed, let

$$\kappa(\tau) = -\frac{1}{\alpha(x,s)}\frac{dw}{d\tau}\frac{2(1-x)^2}{4-3x+s}, \tag{38}$$

and observe that κ is positive and locally integrable by (A1) and (A2). Next define

$$\beta = \int \kappa \, d\tau, \tag{39}$$

and

$$r = e^{\beta/2}. \tag{40}$$

and define the metric coefficients by

$$\lambda = -\frac{1}{2}\log(1-x) \tag{41}$$

$$\mu = -\int \frac{x+y}{4(1-x)}\kappa\, d\tau. \tag{42}$$

It is straightforward to check that λ and μ solve the Einstein equations (5) and (6). The definition of κ now implies

$$\dot{w} = \frac{1}{\kappa}\frac{dw}{d\tau} = -\frac{4-3x+s}{2(1-x)^2}\alpha(x,s), \tag{43}$$

where we recall that dots denote differentiation with respect to $\beta = 2\log r$. Using (27) we thus have

$$(3x+s-2)\dot{x} + 2(1-x)\dot{s} = -\frac{\alpha(x,s)}{2}, \tag{44}$$

which is equivalent to the relation $p + 2p_T = \rho$ in view of (28). We will now show that such a curve exists. Let us fix some small $\epsilon > 0$ and define

$$w_\epsilon(x,s) := \frac{((3-3\epsilon)(1-x)+1+s))^2}{1-x}. \tag{45}$$

Consider now the curve γ_ϵ in the (x,s)-plane defined by

$$w_\epsilon(x,s) = (\epsilon\sqrt{1+3x} + 4(1-\epsilon))^2. \tag{46}$$

Define the corresponding curve in $\mathbb{R}^3$ by

$$(x,y,z) = (x, \max(0,s), \max(0,-s)), \tag{47}$$

so that $s = y - z$. In figure 1 the curve γ_ϵ is depicted together with the curves γ_0 and γ_1. Note that γ_0 is the curve $w(x,s) = 16$ which passes through $(0,0)$ and $(1,-1)$. The curve γ_1 is the curve $\alpha(x,s) = 0$ which also passes through $(0,0)$ and $(1,-1)$. The dotted line shows the line $s = s_q := -z_q$ (for the choice $z_q = 0.6$) and the intersection of the curve γ_0 with this line is the point (x_q, s_q). It is clear that for a sufficiently small $\epsilon > 0$ the curve γ_ϵ intersects an arbitrarily small neighbourhood of (x_q, s_q). Let us denote the

point of intersection of γ_ϵ and the line $s = s_q$ by (x_q^ϵ, s_q). Let us now define $\Gamma := \gamma_\epsilon + h_\epsilon$, where h_ϵ is the curve given by the equation

$$\frac{ds}{dx} = \frac{2s}{s+x}, \text{ such that } s(x_q^\epsilon) = s_q. \tag{48}$$

It is clear from the defining equation that $h_\epsilon \in \{(x,s) : x \geq 0, s \leq 0, s + x > 0\}$ and that the solutions approach the point $(0,0)$ for all admissible starting points (x_q^ϵ, s_q) (note that $x_q + s_q > 0$). The curve h_ϵ is depicted in figure 1.

It remains to show that (A1)-(A5) are satisfied for the curve Γ and that ρ, p and p_T are non-negative along the curve. It is obvious that Γ satisfies (A2)-(A5). To see that it satisfies (A1) we first consider the first part of the curve γ_ϵ and note that $\alpha > 0$ along γ_ϵ. This follows since γ_ϵ lies above γ_1 and $\alpha = 0$ along γ_1 and

$$\frac{\partial \alpha}{\partial s} = 2s + 2 > 0, \text{ for } s > -1.$$

Hence it is sufficient to show that $dw/d\tau < 0$ to establish that

$$\frac{1}{\alpha}\frac{dw}{d\tau} < 0.$$

We differentiate (46) and obtain

$$\frac{dw_\epsilon}{d\tau} = \frac{3\epsilon}{\sqrt{1+3x}} \frac{3(1-\epsilon)(1-x) + 1 + s}{\sqrt{1-x}} \frac{dx}{d\tau}. \tag{49}$$

If we now differentiate (45) directly we get

$$\frac{dw_\epsilon}{d\tau} = \frac{(3-3\epsilon)(1-x)+1+s}{(1-x)^2}\big[(-(3-3\epsilon)(1-x)+1+s)\frac{dx}{d\tau} + 2(1-x)\frac{ds}{d\tau}\big]. \tag{50}$$

Comparing (49) and (50) gives

$$2(1-x)\frac{ds}{d\tau} = \big(\frac{3\epsilon(1-x)^{3/2}}{\sqrt{1+3x}} + 3(1-\epsilon)(1-x) - 1 - s\big)\frac{dx}{d\tau}. \tag{51}$$

Thus differentiating w along γ_ϵ, substituting for $ds/d\tau$ using (51), leads to

$$\frac{dw}{d\tau} = \frac{3\epsilon(4-3x+s)}{1-x} \frac{\sqrt{1-x} - \sqrt{1+3x}}{\sqrt{1+3x}} \frac{dx}{d\tau}. \tag{52}$$

Since $dx/d\tau > 0$ along γ_ϵ and since $0 \leq x < 1$ and $s > -1$ we get that

$$\frac{dw}{d\tau} < 0.$$

It remains to show that $\alpha^{-1}dw/d\tau$ is negative also along the curve h_ϵ. Here we have

$$\begin{aligned}\frac{1}{\alpha}\frac{dw}{d\tau} &= \frac{3(1-x)+1+s}{(x(3x-2)+s(s+2))(1-x)^2}\big[(3x-2+s)+2(1-x)\frac{ds}{dx}\big]\frac{dx}{d\tau}\\ &= \frac{[3(1-x)+1+s]}{(x+s)(1-x)^2}\frac{dx}{d\tau},\end{aligned} \tag{53}$$

where we used (48) for ds/dx. Since $dx/d\tau < 0$ along h_ϵ the claim follows since $x+s>0$ along h_ϵ. Hence,

$$\frac{1}{\alpha}\frac{dw}{\tau} < 0, \tag{54}$$

along Γ. The local integrability of this expression follows by inspection of the formulas above since $0 \le x \le x_q^\epsilon < 1$. Thus condition (A1) holds along Γ. Finally we show that ρ, p and p_T are non-negative along Γ. Since $y \ge 0$ along Γ, cf. (47), it immediately follows that $p \ge 0$. Since (44) implies that $\rho = p + 2p_T$ we only need to show that $p_T \ge 0$ along Γ. First we consider the first part of Γ, i.e., the curve γ_ϵ. From Lemma 1 and (51) we have

$$\begin{aligned}8\pi r^2 p_T &= \frac{x+s}{2(1-x)}\dot{x} + \dot{s} - z + \frac{(x+s)^2}{4(1-x)}\\ &= \big(\frac{3\epsilon\sqrt{1-x}}{2\sqrt{1+3x}} + 1 - \frac{3}{2}\epsilon\big)\frac{1}{\kappa}\frac{dx}{d\tau} + [-z + \frac{(x+s)^2}{4(1-x)}].\end{aligned} \tag{55}$$

Since ϵ is small the first term is positive since $dx/d\tau > 0$ along γ_ϵ. The term in square brackets can also be seen to be positive. Indeed, along the part of γ_ϵ where $s \ge 0$, $z = 0$ and the claim is trivial so we focus on the part where $s < 0$. Here $z = -s$ and we thus want to show that

$$s + \frac{(x+s)^2}{4(1-x)} > 0 \tag{56}$$

along the part of γ_ϵ where $s < 0$. We evaluate the left hand side of (56) along γ_0 and show that it is positive there and then we conclude by continuity that this statement also holds along γ_ϵ for ϵ small. Along γ_0 we have the relation

$$x = \frac{4+3s}{9} + \frac{4}{3}\sqrt{\frac{1}{9} - \frac{s}{3}}. \tag{57}$$

A straightforward calculation now gives that along γ_0

$$s + \frac{(x+s)^2}{4(1-x)} = s + 4(\frac{1}{3} + \sqrt{\frac{1}{9} - \frac{s}{3}})^2 \ge s + \frac{16}{9} > 0.$$

Hence $p_T > 0$ also along γ_ϵ for a sufficiently small ϵ. Along h_ϵ it holds by construction, cf. (48) and Lemma (1), that $\rho = 0$ and thus $p = p_T = 0$. This completes the proof of Theorem 1.

$\square$

Proof of Theorem 2. Let us begin with a few general facts. Consider a regular solution where the matter quantities are supported in $[0, R]$. We recall from section 2 the following consequence of the matching condition

$$e^{-\lambda(r)} = e^{\mu(r)} = \sqrt{1 - \frac{2M}{r} + \frac{Q^2(r)}{r^2}},\ r \geq R, \tag{58}$$

so that $e^{\mu+\lambda} = 1$ for $r > R$. Let us now derive an explicit expression for μ. The Einstein equation (6) can be written as

$$\mu_r = (\frac{m_i}{r^2} + 4\pi r p - \frac{q^2}{2r^3} + \frac{F}{2r^2})e^{2\lambda}, \tag{59}$$

so that

$$\mu(r) = -\int_r^\infty (\frac{m_i}{r^2} + 4\pi r p - \frac{q^2}{2r^3} + \frac{F}{2r^2})e^{2\lambda} dr, \tag{60}$$

since $\mu \to 0$ as $r \to \infty$ in view of (58). We will also need an explicit formula for λ_r, and from (5) we have

$$\lambda_r = (4\pi r \rho(r) - \frac{m_i(r)}{r^2} + \frac{q^2}{2r^3} - \frac{F}{2r^2})e^{2\lambda}. \tag{61}$$

From the expressions of μ_r and λ_r we also obtain

$$\mu(r) + \lambda(r) = -\int_r^\infty 4\pi\eta(\rho + p)e^{2\lambda} d\eta, \tag{62}$$

so that in particular

$$\mu + \lambda \leq 0. \tag{63}$$

Now we derive our fundamental integral equation which is a consequence of the Tolman-Oppenheimer-Volkov equation. Let

$$\psi = (m_g + 4\pi r^3 p - \frac{q^2}{r})e^{\mu+\lambda}.$$

Taking the derivative of ψ with respect to r, a straightforward calculation using the Tolman-Oppenheimer-Volkov equation (11) results in the following equation

$$(m_g + 4\pi r^3 p - \frac{q^2}{r})e^{\mu+\lambda} = \int_0^r e^{\mu+\lambda}(4\pi\eta^2(\rho + p + 2p_T) + \frac{q^2}{\eta^2})d\eta. \tag{64}$$

This equation must be satisfied by any spherically symmetric static solution of the Einstein-Maxwell system.

Let us now consider our sequence of solutions. Since $p_k(R) = 0,\ (m_g)_k(R) = M_k,\ q_k(R) = Q_k$ and $e^{\mu_k+\lambda_k}(R) = 1$ we get in view of (64) for $r = R$,

$$M_k - \frac{Q_k^2}{R} = \int_{R_k}^{R} e^{\mu_k+\lambda_k}(4\pi\eta^2(\rho_k + p_k + 2(p_T)_k) + \frac{q_k^2}{\eta^2})\, d\eta. \tag{65}$$

Here we also used the fact that the matter is supported in $[R_k, R]$. We split the right hand side as follows

$$\begin{aligned} &\int_{R_k}^{R} e^{\mu_k+\lambda_k}(4\pi\eta^2(\rho_k + p_k + 2(p_T)_k) + \frac{q_k^2}{\eta^2})\, d\eta \\ &= \int_{R_k}^{R} e^{\mu_k+\lambda_k}(8\pi\eta^2\rho_k + \frac{q_k^2}{\eta^2})\, d\eta \\ &\quad + \int_{R_k}^{R} e^{\mu_k+\lambda_k}(4\pi\eta^2(p_k + 2(p_T)_k - \rho_k)\, d\eta =: S_k + T_k. \end{aligned} \tag{66}$$

By the mean value theorem we get that there is a $\xi \in [R_k, R]$ such that

$$S_k = 2e^{\mu_k(\xi)}\xi \int_{R_k}^{R} e^{\lambda_k}(4\pi\eta\rho_k + \frac{q_k^2}{2\eta^3})d\eta \tag{67}$$

$$= 2e^{\mu_k(\xi)}\xi \int_{R_k}^{R} [-\frac{d(e^{-\lambda_k})}{d\eta}]d\eta + 2e^{\mu_k(\xi)}\xi \int_{R_k}^{R} (\frac{(m_i)_k(\eta)}{\eta^2} + \frac{F_k(\eta)}{2\eta^2})e^{\lambda_k}d\eta$$

$$=: S_k^1 + S_k^2. \tag{68}$$

Here we used equation (61) for λ_r. Now, since $\sup_k q_k/r$ is strictly less than one we obtain a uniform bound on λ_k from the inequality (1), cf. the consistency check after the formulation of Theorem 1. The same computation guarantees that $(m_i)_k(r)/r + F_k(r)/2r < 1/2$, thus it follows that

$$0 \le S_k^2 \le C \log\big(\frac{R}{R_k}\big) \to 0 \text{ as } k \to \infty.$$

Since $\mu_k + \lambda_k \le 0$ by (63) and since $\rho_k \ge p_k + 2(p_T)_k \ge 2(p_T)_k$, it follows from the assumptions on the sequence that also

$$T_k \to 0 \text{ as } k \to \infty.$$

For the term S_k^1 we get by (58)

$$S_k^1 = -2e^{\mu_k(\xi)}\xi \int_{R_k}^{R} \frac{d}{d\eta}(e^{-\lambda_k})\, d\eta = 2e^{\mu_k(\xi)}\xi\big(1 - \sqrt{1 - \frac{2M_k}{R} + \frac{Q_k^2}{R^2}}\,\big).$$

Note here that $\lambda_k(R_k) = 0$ due to the support condition of the matter terms. In view of (60) and the general bounds on m_i/r and q/r it follows that

$$e^{\mu_k(\xi)} \to \sqrt{1 - \frac{2M}{R} + \frac{Q^2}{R^2}} \text{ as } k \to \infty,$$

so that

$$\lim_{k\to\infty} S_k^1 = 2R\sqrt{1 - \frac{2M}{R} + \frac{Q^2}{R^2}}\big(1 - \sqrt{1 - \frac{2M}{R} + \frac{Q^2}{R^2}}\,\big).$$

In conclusion, from (65) we get in the limit $k \to \infty$,

$$M - \frac{Q^2}{R} = 2R\sqrt{1 - \frac{2M}{R} + \frac{Q^2}{R^2}}\big(1 - \sqrt{1 - \frac{2M}{R} + \frac{Q^2}{R^2}}\,\big). \qquad (69)$$

After some algebra this relation can be written as

$$M - \frac{Q^2}{R} = (3M - \frac{Q^2}{R})\sqrt{1 - 2M/R + \frac{Q^2}{R^2}}.$$

Squaring both sides one finds after some rearrangements

$$(9M^2 - \frac{6MQ^2}{R} + \frac{Q^4}{R^2})(\frac{2M}{R} - \frac{Q^2}{R^2}) = 4MR(\frac{2M}{R} - \frac{Q^2}{R^2}),$$

so that

$$(3M - \frac{Q^2}{R})^2 = 4MR. \qquad (70)$$

We have thus arrived at the same expression (with equality instead of inequality) as (35) and we accordingly obtain

$$\sqrt{M} = \frac{\sqrt{R}}{3} + \sqrt{\frac{R}{9} + \frac{Q^2}{3R}},$$

which completes the proof of Theorem 2.

□

5 Final remarks

In [18] several different conditions on the relation between ρ, p and p_T are investigated, e.g. the isotropic case where $p = p_T$. We have not tried to consider other cases than $p + p_T \leq \rho$ in this work although we believe that it

can be done. We believe however that an equally transparent inequality as (1) is unlikely to be found under other conditions than $p + p_T \leq \rho$, cf. [18]. However, the following comparison with the non-charged case is interesting. The original Buchdahl inequality [10] was derived under the assumptions that ρ is non-increasing outwards and the pressure is isotropic and the steady state that saturates the inequality $2M/R \leq 8/9$ within this class of solutions is the one with constant energy density for which the pressure is infinite at the center. It is quite remarkable that exactly the same inequality holds much more generally [1], as long as $p + p_T \leq \rho$, and in particular that the steady state that saturates the inequality in this class is an infinitely thin shell which is drastically different from the constant energy density solution. One can now ask if there is a similar analogue in the charged case.

In the work [16] by Giuliani and Rothman they find an explicit solution with constant energy density and constant charge density and they obtain for this solution an algebraic equation from which the values of the stability radius can be evaluated. It is in view of the discussion above therefore interesting to see whether these values are less, equal or greater than the values given by (1). In [16] the ratios R/M are displayed for different ratios of Q/R (or more precisely for different ratios Q/M, but the corresponding ratios Q/M can be deduced). It turns out that the critical stability radius given by the relation

$$\sqrt{M} = \frac{1}{3} + \sqrt{\frac{1}{9} + \frac{Q^2}{3R}}$$

are smaller than the corresponding ones found in [16], or alternatively, our relation admits a larger ratio M/R for a given ratio Q/M.

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